Note on the Value of the Longitude in the Lunar Theory when the Sun's Mass is put Zero. By P. H. Cowell, B.A., Fellow of Trinity College, and Isaac Newton, Student in the University of Cambridge.

(*Communicated by F. W. Dyson, M.A.*)

About fifteen years ago a paper was published in Silliman's journal* by Mr. Stockwell, in which he called attention to the fact that when the mass of the Sun, and therefore its mean motion, is put equal to zero, Delaunay's expression for the longitude † does not degenerate into the ordinary formula for elliptic motion, but contains terms in which the Sun's eccentricity, and the position of the Sun's apse, and the ratio of the parallaxes enter. Mr. Stockwell argued that there must therefore be "something seriously wrong" with Delaunay's analysis.

Mr. Stockwell's objection has been discussed by M. Gogou.‡ He has shown that when the Sun's mass is zero, and consequently when the Moon's node and apse are fixed, certain operations of Delaunay's may be modified. Some terms with different arguments coalesce, and the accuracy of Delaunay's results is fully established, so far as the terms objected to by Mr. Stockwell are concerned.

M. Gogou's work, which involves a modification of several of Delaunay's operations, is very long and laborious. Moreover it merely verifies Delaunay's terms without explaining the physical (in contradistinction to the analytical) reason why such terms should exist. In this paper the question is treated from a different standpoint; the mean motion of the Sun, instead of being put zero, is made to diminish without limit.

At the beginning of his analysis Delaunay replaced the mass of the Sun m' by $n'^2a'^3$ (his notation is well known, and it is unnecessary to explain the meaning of the symbols that occur in this paper); whereas, more accurately, the relation is

$$\mu + m' = n'^2a'^3,$$

μ being the mass of the Earth. Delaunay's analysis is therefore founded upon the assumption that the Sun's mass is extremely large in comparison with the Earth's. Now, when we suppose n' to be diminished to a very small fraction of its actual value, we must either suppose a' increased or m' diminished. It is inconvenient to suppose a' increased, as it would diminish some of the terms in Delaunay's degenerate expression that we are about to consider, and an explanation of them as they stand is more

* *American Journal*, vol. xx. 3rd series, 1880.

† *Mémoires de l'Académie des Sciences*, vol. xxix. ch. xi.

‡ *Annales de l'Observatoire de Paris*, vol. xviii. 1885.

satisfactory. On the other hand, if m' is greatly diminished, the Earth's mass may not be neglected in comparison with it. Now Delaunay's final expression for the longitude has been calculated to the seventh order of small quantities. The quantities neglected, being of the eighth order, are comparable with 10^{-8} or 10^{-9}, whereas the ratio $\mu : m'$ is apparently 3×10^{-6}. Again, if E be the mass of the Earth, M the mass of the Moon, Delaunay neglects M/E. We know, however, that if we replace $1/a'$ by $(E-M)/(E+M)a'$ wherever it occurs in Delaunay's final expression, the parallactic terms that contain the first power of the ratio of the parallaxes are by this change rendered perfectly accurate, but the terms containing the square and higher powers of the ratio of the parallaxes are not rendered accurate in this manner, though the error is very small, and probably is comparable with 10^{-7}. A physical interpretation must be given to the simplifications (such as neglecting μ/m') introduced by Delaunay, and then his results become the solution, accurate to the seventh order, of a definite problem, different—but not very different—from the problem actually presented by the Moon.

We are led, therefore, to examine what physical problem Delaunay has actually solved.

In the problem of two bodies, the Earth and Sun, suppose we have the relation

$$\mu + m' = n'^2 a'^3.$$

If, however, one of the two bodies be supposed fixed (in the same sense that the centre of gravity of two or more free bodies may be supposed fixed), we may conceive that one body attracts the other, but is not attracted by it, or else that external forces are called into play, neutralising the attraction of the second body upon the first; the relation then becomes

$$m' = n'^2 a'^3,$$

which is that used by Delaunay. Delaunay, therefore, solves the following problem—to investigate the motion, supposed nearly circular, of a Moon of no mass under the attraction of the Earth and disturbed by a distant Sun, which produces acceleration in both the Earth and Moon according to Newton's law, but which is not itself affected by attractions from them.

This is a definite problem, and one that reduces to elliptic motion when the Sun's mass vanishes. Moreover, when the Sun's mass is large the error in supposing the Sun at rest instead of the centre of gravity at rest is not large, although larger than some of the terms given by Delaunay. When the Sun's mass is small the problem ceases to have any resemblance to that of three free bodies.

Let us suppose the Sun's mass so small that n'/n is of the seventh order of small quantities. Then no periodic term in the

solution due to the action of the Sun is of a lower order than the eighth, and the motion must therefore be sensibly elliptic. However, the node and apse move continually in one direction, although with extreme slowness, so that after the lapse of a long time the ellipse described is a distinctly different one. Analysis also shows that the eccentricity and inclination undergo finite variations (that is to say, variations that do not become insensible when m is insensible). The eccentricity and inclination at any instant turn out to be functions of the configuration of the node and apses, and are therefore only constant when the configuration remains unchanged, that is to say, when m' and n' are absolutely zero.

The explanation of the terms discussed by Mr. Stockwell and M. Gogou, therefore, is that, when terms containing m are neglected, Delaunay's expressions must represent, not merely elliptic motion in one definite and fixed ellipse, but must be capable of representing motion in any one of that doubly infinite variety of ellipses, with which the sensible motion coincides, when any arbitrary values are given to the longitudes of apse and node.

It remains to verify the foregoing explanation by showing that by formulæ of transformation that involve the configuration of node and apses only Delaunay's degenerate expressions can be identified with the expressions for elliptic motion. Assuming, therefore, the identity of Delaunay's degenerate expression for the longitude with the expression in purely elliptic motion for all values of the time, we may separately equate those terms whose speed is the same multiple of the mean motion. We shall therefore obtain the relations we seek from the terms in the longitude whose speed is the mean motion or twice the mean motion, and we shall verify the algebraic work by a comparison of those terms whose speed is three times the mean motion. This verification will also constitute a sufficient guarantee of the accuracy of the foregoing explanation as to the reasons why the terms under discussion appear. It appears scarcely necessary to perform the corresponding algebraic work for the terms in the latitude and parallax and the terms of shorter period in the longitude.

We give below those terms in Delaunay's expression for the longitude that are independent of m, and whose speed is once, twice, or thrice the mean motion. We restore Delaunay's notation of l, g, h, g', h' (in chapter xi. he has used the symbols D, F, l), and we write χ for $h+g-h'-g'$, the distance between the apses, and we notice that there are no terms containing l'. This last fact is a confirmation of the theory, because there are no terms of corresponding speed in the purely elliptic expression. The expression referred to is

$$\left(2e-\frac{1}{4}e^3+\frac{5}{96}e^5\right)\sin l$$

$$+\left(-3\gamma^2e-18\gamma^4e+\frac{61}{8}\gamma^2e^3-\frac{447}{4}\gamma^6e+92\gamma^4e^3-\frac{925}{96}\gamma^2e^5-\frac{165}{16}\gamma^2e\frac{a^2}{a'^2}\right)\sin(2g+l)$$

$$+\left(\frac{7}{6}\gamma^2e^3+\frac{359}{12}\gamma^4e^3-\frac{285}{64}\gamma^2e^5\right)\sin(2g-l)$$

$$+\frac{11}{24}\gamma^4e^3\sin(4g+l)$$

$$+\frac{165}{64}ee'^2\frac{a^2}{a'^2}\sin(2\chi+l)$$

$$+\left(\frac{5}{2}e'-\frac{15}{2}\gamma^2e'+\frac{15}{2}e^2e'+\frac{5}{2}e'^3+\frac{15}{2}\gamma^4e'-\frac{55}{2}\gamma^2e^2e'+\frac{465}{128}e^4e'\right)\frac{a}{a'}\sin(\chi+l)$$

$$+\left(\frac{105}{16}e^2e'-\frac{365}{16}\gamma^2e^2e'+\frac{385}{96}e^4e'\right)\frac{a}{a'}\sin(\chi-l)$$

$$+\frac{205}{16}\gamma^2e^2e'\frac{a}{a'}\sin(\chi+2g+l)$$

$$+\left(\frac{5}{6}\gamma^2e'+\frac{5}{3}\gamma^4e'+\frac{145}{48}\gamma^2e^2e'\right)\frac{a}{a'}\sin(\chi-2g-l)-\frac{115}{144}\gamma^2e^2e'\frac{a}{a'}\sin(\chi-2g+l)$$

$$+\left(\frac{5}{4}e^2-\frac{5}{4}\gamma^2e^2-\frac{11}{24}e^4-\frac{85}{8}\gamma^4e^2+\frac{35}{16}\gamma^2e^4+\frac{17}{192}e^6\right)\sin 2l$$

$$+\left(-\gamma^2-\gamma^4-\frac{9}{4}\gamma^2e^2-\gamma^6-\frac{89}{4}\gamma^4e^2+\frac{165}{16}\gamma^2e^4\right)\sin(2g+2l)$$

$$+\frac{99}{64}\gamma^2e^4\sin(2g-2l)+\frac{39}{16}\gamma^4e^2\sin(4g+2l)$$

$$+\frac{125}{64}e'^2\frac{a^2}{a'^2}\sin(2\chi+2l)+\left(\frac{25}{8}ee'-\frac{265}{24}\gamma^2ee'+\frac{135}{16}e^3e'+\frac{25}{8}ee'^3\right)\frac{a}{a'}\sin(\chi+2l)$$

$$+\frac{805}{96}e^3e'\frac{a}{a'}\sin(\chi-2l)-\frac{75}{16}\gamma^2ee'\frac{a}{a'}\sin(\chi+2g+2l)-\frac{35}{24}\gamma^2ee'\frac{a}{a'}\sin(\chi-2g-2l)$$

$$+\left(\frac{13}{12}e^3-\frac{5}{2}\gamma^2e^3-\frac{43}{64}e^5-\frac{165}{8}\gamma^4e^3+5\gamma^2e^5\right)\sin 3l$$

$$+\left(-2\gamma^2e-2\gamma^4e-\frac{11}{8}\gamma^2e^3-2\gamma^6e-\frac{49}{2}\gamma^4e^3+\frac{881}{64}\gamma^2e^5\right)\sin(2g+3l)$$

$$+\frac{1357}{640}\gamma^2e^5\sin(2g-3l)$$

$$+\left(3\gamma^4e+21\gamma^6e-\frac{105}{16}\gamma^4e^3\right)\sin(4g+3l)$$

$$+\frac{325}{64}ee'^2\frac{a^2}{a'^2}\sin(2\chi+3l)$$

$$+\left(\frac{65}{16}e^2e'-\frac{895}{48}\gamma^2e^2e'+\frac{315}{32}e^4e'\right)\frac{a}{a'}\sin(\chi+3l)$$

$$+\frac{2815}{256}e^4e'\frac{a}{a'}\sin(\chi-3l)$$

$$+\left(-\frac{5}{2}\gamma^2e'+5\gamma^4e'-\frac{245}{16}\gamma^2e^2e'\right)\frac{a}{a'}\sin(\chi+2g+3l)-\frac{505}{48}\gamma^2e^2e'\frac{a}{a'}$$

$$\sin(\chi-2g-3l)-\frac{5}{6}\gamma^4e'\frac{a}{a'}\sin(\chi-4g-3l).$$

The expression in purely elliptic motion is

$$\left(2e_0-\frac{1}{4}e_0^3+\frac{5}{96}e_0^5\right)\sin l_0+\left(2\gamma_0^2e_0+2\gamma_0^4e_0-\frac{7}{4}\gamma_0^2e_0^3+2\gamma_0^6e_0\right.$$

$$\left.-\frac{7}{4}\gamma_0^4e_0^3+\frac{5}{96}\gamma_0^2e_0^5\right)\sin(2g_0+l_0)$$

$$+\left(-\frac{1}{12}\gamma_0^2e_0^3-\frac{1}{12}\gamma_0^4e_0^3-\frac{5}{192}\gamma_0^2e_0^5\right)\sin(2g_0-l_0)-\frac{17}{12}\gamma_0^4e_0^3\sin(4g_0+l_0)$$

$$+\left(\frac{5}{4}e_0^2-\frac{11}{24}e_0^4+\frac{17}{192}e_0^6\right)\sin 2l_0+\left(-\gamma_0^2-\gamma_0^4+4\gamma_0^2e_0^2-\gamma_0^6+4\gamma_0^4e_0^2\right.$$

$$\left.-\frac{55}{16}\gamma_0^2e_0^4\right)\sin(2g_0+2l_0)-\frac{1}{24}\gamma_0^2e_0^4\sin(2g_0-2l_0)+\frac{11}{4}\gamma_0^4e_0^2$$

$$\sin(4g_0+2l_0)$$

$$+\left(\frac{13}{12}e_0^3-\frac{43}{64}e_0^5\right)\sin 3l_0+\left(-2\gamma_0^2e_0-2\gamma_0^4e_0+\frac{27}{4}\gamma_0^2e_0^3-2\gamma_0^6e_0+\frac{27}{4}\gamma_0^4e_0^3\right.$$

$$\left.-\frac{207}{32}\gamma_0^2e_0^5\right)\sin(2g_0+3l_0)-\frac{9}{320}\gamma_0^2e_0^5\sin(2g_0-3l_0)+\left(-2\gamma_0^4e_0\right.$$

$$\left.-4\gamma_0^6e_0+\frac{45}{4}\gamma_0^4e_0^3\right)\sin(4g_0+3l_0).$$

where only the terms of the same speed as before are retained. A suffix zero has been placed after e_0, γ_0, g_0, l_0 in this expression, $l-l_0$ is constant and small, but there is no reason why it should be zero.

Beginning with the terms whose speed is the mean motion and retaining terms of the third order only, we get

$$\left(2e-\frac{1}{4}e^3\right)\sin l-3\gamma^2e\sin(2g+l)+\frac{5}{2}e'\frac{a}{a'}\sin(\chi+$$

$$=\left(2e_0-\frac{1}{4}e_0^3\right)\sin l_0+2\gamma_0^2e_0\sin(2g_0+l$$

As a first approximation

$$e\sin l=e_0\sin l_0.$$

In time $\pi/2n$, we have

$$e\sin(l+\pi/2)=e_0\sin(l_0+\pi/2)$$

or

$$e\cos l=e_0\cos l_0.$$

As a general rule we may in any equation change sines into cosines, changing the sign of the coefficient, however, when the coefficient of l or l_0 is negative.

On referring to the equation obtained by equating terms containing $2l$, $2l_0$, we find that we commit an error of the fourth order in replacing

$$\gamma_0^2 \sin(2g_0 + 2l_0)$$

by

$$\gamma^2 \sin(2g + 2l)$$

It follows that $e_0 - e$, $\gamma_0^2 - \gamma^2$, $l_0 - l$, $g_0 - g$, are respectively of the third, fourth, second and second orders, and hence in any equation we may drop the suffixes in the terms of the highest order retained.

We therefore at once obtain as a second approximation

$$e_0 \sin l_0 = e \sin l - \frac{5}{2}\gamma^2 e \sin(2g + l) + \frac{5}{4} e' \frac{a}{a'} \sin(\chi + l) \quad . \quad . \quad . \quad (1)$$

Again, from the terms whose speed is equal to twice the mean motion, we get correctly to the fourth order.

$$\left(\frac{5}{4}e^2 - \frac{5}{4}\gamma^2 e^2 - \frac{11}{24}e^4\right)\sin 2l + \left(-\gamma^2 - \gamma^4 - \frac{9}{4}\gamma^2 e^2\right)\sin(2g + 2l)$$

$$+\frac{25}{8}ee'\frac{a}{a'}\sin(\chi + l) = \left(\frac{5}{4}e_0^2 - \frac{11}{24}e_0^4\right)\sin 2l_0 + (-\gamma_0^2 - \gamma_0^4 + 4\gamma_0^2 e_0^2)\sin(2g_0 + 2l_0)$$

We deduce from (1) and the corresponding formula where cosines replace sines,

$$e_0^2 \sin 2l_0 = e^2 \sin 2l - 5\gamma^2 e^2 \sin(2g + 2l) + \frac{5}{2}ee'\frac{a}{a'}\sin(\chi + 2l)$$

Whence

$$\gamma_0^2 \sin(2g_0 + 2l_0) = \gamma^2 \sin(2g + 2l) + \frac{5}{4}\gamma^2 e^2 \sin 2l \quad . \quad . \quad . \quad (2)$$

From (1) we deduce

$$e_0^2 = e^2 - 5\gamma^2 e^2 \cos 2g + \frac{5}{2}ee'\frac{a}{a'}\cos\chi$$

and from (2),

$$\gamma_0^2 = \gamma^2 + \frac{5}{4}\gamma^2 e^2 \cos 2g$$

We must now return to the first equation and retain the terms of the fifth order, and approximate to the terms of the third order as far as the fifth order, by means of the results already obtained. We then return to the second equation and approximate as far as the sixth order. We thus obtain,

$$e_0 \sin l_0 = e \sin l - \frac{5}{2}\gamma^2 e \sin(2g + l)$$

$$+\frac{5}{4}e'\frac{a}{a'}\sin(\chi+l)$$

$$+\left(\frac{5}{2}\gamma^4 e-\frac{5}{4}\gamma^2 e^3\right)\sin l$$

$$+\left(-10\gamma^4 e+\frac{65}{16}\gamma^2 e^3\right)\sin(2g+l)$$

$$+\frac{15}{16}\gamma^2 e^3\sin(2g-l)$$

$$+\left(-\frac{15}{4}\gamma^2 e'+\frac{65}{16}e^2 e'+\frac{5}{4}e'^3\right)\frac{a}{a'}\sin(\chi+l)$$

$$+\frac{25}{8}e^2 e'\frac{a}{a'}\sin(\chi-l)+\frac{5}{3}\gamma^2 e'\frac{a}{a'}\sin(\chi-2g-l)$$

and

$$\gamma_0^2\sin(2g_0+2l_0)=\gamma^2\sin(2g+2l)+\frac{5}{4}\gamma^2 e^2\sin 2l$$

$$+\left(5\gamma^4 e^2-\frac{5}{16}\gamma^2 e^4\right)\sin 2l+\left(\frac{5}{4}\gamma^4 e^2-\frac{5}{32}\gamma^2 e^4\right)\sin(2g+2l)$$

$$-\frac{25}{64}\gamma^2 e^4\sin(2g-2l)-\frac{5}{2}\gamma^4 e^2\sin(4g+2l)$$

$$+\frac{5}{3}\gamma^2 ee'\frac{a}{a'}\sin(\chi+2l)+\frac{15}{8}\gamma^2 ee'\frac{a}{a'}\sin(\chi+2g+2l)$$

$$+\frac{5}{8}\gamma^2 ee'\frac{a}{a'}\sin(\chi-2g-2l).$$

In both of which equations we may replace sines by cosines, changing signs where necessary. As a verification, we substitute these values in the terms of the elliptic expression, whose speed is three times the mean motion. These terms are,

$$\frac{13}{12}e_0^3\sin 3l_0-2\gamma_0^2 e_0\sin(2g_0+3l_0)-\frac{43}{64}e_0^5\sin 3l_0$$

$$+\left(-2\gamma_0^4 e_0+\frac{27}{4}\gamma_0^2 e_0^3\right)\sin(2g_0+3l_0)$$

$$-2\gamma_0^4 e_0\sin(4g_0+3l_0)$$

$$+\left(-2\gamma_0^6 e_0+\frac{27}{4}\gamma_0^4 e_0^3-\frac{207}{32}\gamma_0^2 e_0^5\right)\sin(2g_0+3l_0)$$

$$-\frac{9}{320}\gamma_0^2 e_0^5\sin(2g_0-3l_0)+\left(-4\gamma_0^6 e_0+\frac{45}{4}\gamma_0^4 e_0^3\right)\sin(4g_0+3l_0).$$

They become, after the substitutions have been performed,

$$\frac{13}{12}e^3\sin 3l-2\gamma^2 e\sin(2g+3l)$$

$$+\left(-\frac{5}{2}\gamma^2 e^3-\frac{43}{64}e^5\right)\sin 3l$$

$$+\left(-2\gamma^4 e - \frac{11}{8}\gamma^2 e^3\right)\sin(2g+3l)$$

$$+3\gamma^4 e\sin(4g+3l)$$

$$+\frac{65}{16}e^2 e'\frac{a}{a'}\sin(\chi+3l)-\frac{5}{2}\gamma^2 e'\frac{a}{a'}\sin(\chi+2g+3l)$$

$$+\left(-\frac{165}{8}\gamma^4 e^3+5\gamma^2 e^5\right)\sin 3l$$

$$+\left(-2\gamma^6 e-\frac{49}{2}\gamma^4 e^3+\frac{881}{64}\gamma^2 e^5\right)\sin(2g+3l)$$

$$+\frac{1357}{640}\gamma^2 e^5\sin(2g-3l)$$

$$+\left(21\gamma^6 e-\frac{105}{16}\gamma^4 e^3\right)\sin(4g+3l)$$

$$+\frac{325}{64}ee'^2\left(\frac{a}{a'}\right)^2\sin(2\chi+3l)$$

$$+\left(-\frac{895}{48}\gamma^2 e^2 e'+\frac{315}{32}e^4 e'+\frac{65}{16}e^2 e'^3\right)\frac{a}{a'}\sin(\chi+3l)$$

$$+\frac{2815}{256}e^4 e'\frac{a}{a'}\sin(\chi-3l)$$

$$+\left(5\gamma^4 e'-\frac{245}{16}\gamma^2 e^2 e'-\frac{5}{2}\gamma^2 e'^3\right)\frac{a}{a'}\sin(\chi+2g+3l)$$

$$-\frac{505}{48}\gamma^2 e^2 e'\frac{a}{a'}\sin(\chi-2g-3l)-\frac{5}{6}\gamma^4 e'\frac{a}{a'}\sin(\chi-4g-3l).$$

Delaunay omits the two terms

$$\frac{65}{16}e^2 e'^3\frac{a}{a'}\sin(\chi+3l)-\frac{5}{2}\gamma^2 e'^3\frac{a}{a'}\sin(\chi+2g+3l),$$

which he considers as being of the eighth order, on account of the smallness of e'. With this exception the expression just obtained agrees with the corresponding terms of Delaunay's expression. This verifies the algebraical work. Squaring and adding the expressions for $e_0 \sin l_0$, $e_0 \cos l_0$.

$$e_0^2 = e^2 - 5\gamma^2 e^2\cos 2g+\frac{5}{2}ee'\frac{a}{a'}\cos\chi$$

$$+\frac{45}{4}\gamma^4 e^2-\frac{5}{2}\gamma^2 e^4+\frac{25}{16}e'^2\frac{a^2}{a'^2}$$

$$+\left(-20\gamma^4 e^2+\frac{25}{4}\gamma^2 e^4\right)\cos 2g$$

$$+\left(-\frac{15}{2}\gamma^2 ee'+\frac{15}{8}e^3 e'+\frac{5}{2}ee'^3\right)\frac{a}{a'}\cos\chi$$

$$-\frac{115}{12}\gamma^2 ee'\frac{a}{a'}\cos(\chi-2g).$$

Again, from the expressions for

$$\gamma_0^2 \sin(2g_0+2l_0),\ \gamma_0^2 \cos(2g_0+2l_0),$$

$$\gamma_0^2 \cos(2g_0+2l_0-2g-2l) = \gamma^2 + \frac{5}{4}\gamma^2 e^2 \cos 2g$$

$$+\left(\frac{5}{4}\gamma^4 e^2 - \frac{5}{32}\gamma^2 e^4\right) + \left(\frac{5}{2}\gamma^4 e^2 - \frac{5}{16}\gamma^2 e^4\right)\cos 2g$$

$$+\frac{25}{64}\gamma^2 e^4 \cos 4g + \frac{5}{4}\gamma^2 ee'\frac{a}{a'}\cos\chi$$

$$+\frac{5}{3}\gamma^2 ee'\frac{a}{a'}\cos(\chi-2g)$$

and

$$\gamma_0^2 \sin(2g_0+2l_0-2g-2l) = -\frac{5}{4}\gamma^2 e^2 \sin 2g$$

$$\therefore \gamma_0^2(1-\cos(2g_0+2l_0-2g-2l)) = \frac{25}{64}\gamma^2 e^4(1-\cos 4g)$$

$$\therefore \gamma_0^2 = \gamma^2 + \frac{5}{4}\gamma^2 e^2 \cos 2g + \left(\frac{5}{4}\gamma^4 e^2 + \frac{15}{64}\gamma^2 e^4\right)$$

$$+\left(\frac{5}{2}\gamma^4 e^2 - \frac{5}{16}\gamma^2 e^4\right)\cos 2g + \frac{5}{4}\gamma^2 ee'\frac{a}{a'}\cos\chi$$

$$+\frac{5}{3}\gamma^2 ee'\frac{a}{a'}\cos(\chi-2g).$$

Correctly, therefore, to the fourth order e_0^2 must lie between

$$e^2 \pm 5\gamma^2 e^2 + 5ee'\frac{a}{a'}$$

and γ_0^2 between

$$\gamma^2 \pm \frac{5}{4}\gamma^2 e^2$$

This result bears some resemblance to planetary theory.

The conditions of the problem would be approximately realised for a small satellite of *Neptune's* revolving near the surface of its primary.

Mean Areas and Heliographic Latitudes of Sun-spots in the year 1893, *deduced from Photographs taken at the Royal Observatory, Greenwich, at Dehra Dûn (India), and in Mauritius.*

(*Communicated by the Astronomer Royal.*)

The results here given are in continuation of those printed in the *Monthly Notices*, vol. lv. p. 150, and are deduced from the measurements of solar photographs taken at the Royal Observatory, Greenwich, at Dehra Dûn, India, and at the Royal Alfred Observatory, Mauritius.